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Properties of the L_p-Solutions of Linear Impulsive Equations in a Banach Space

D. D. Bainov,[1] S. I. Kostadinov,[1] and A. D. Myshkis[2]

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Conditions for the existence of L_p-solutions $(1 \leq p \leq \infty)$ of linear impulsive equations in a Banach space are found.

1. INTRODUCTION

Impulsive equations are useful mathematical models of processes and phenomena in many fields of science and technology. These equations describe evolutionary processes which suddenly change their state at certain moments. In the mathematical simulation of such phenomena the duration of the change of the state is neglected and it is assumed that this change takes place by jumps. Processes of such character are observed in numerous problems of theoretical physics, control theory, biology, the theory of queues, ecology, pharmacokinetics, etc.

The theory of impulsive differential equations was first developed by Myshkis and Mil'man (1960). After this publication numerous papers (e.g., Simeonov and Bainov, 1985*a,b*; Bainov and Simeonov, 1989) devoted to this subject appeared. The interest in impulsive differential equations is explained by the fact that, besides the great possibilities for the simulation of various processes, impulsive equations have many specific characteristics, such as the "beating" of solutions, the merging of solutions, bifurcation, the dying of solutions, the loss of the property of autonomy, etc., which make their theory much richer than the theory of ordinary differential equations.

Together with the other branches of this theory, in recent years the development of the theory of abstract impulsive differential equations has begun (Bainov *et al.*, 1988; Zabreiko *et al.*, 1988).

[1]Plovdiv University "Paissii Hilendarskii".
[2]Moscow Institute of Railway Transport Engineers.

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In the present paper conditions for the existence of solutions of abstract linear impulsive differential equations lying in the space L_p $(1 \le p \le \infty)$ are found. Note that many results obtained in this work are also new for equations without an impulse effect.

2. STATEMENT OF THE PROBLEM

Let X be an arbitrary complex Banach space. Consider the impulsive equations

$$\frac{dx}{dt} = A(t)x|_{t \neq t_n} \tag{1}$$

$$x(t_n^+) = Q_n x(t_n^-) \qquad (n \in \mathbb{Z}) \tag{2}$$

$$\frac{dx}{dt} = A(t)x + f(t)|_{t \neq t_n} \tag{3}$$

$$x(t_n^+) = Q_n x(t_n^-) + h_n \qquad (n \in \mathbb{Z}) \tag{4}$$

where $A: \mathbb{R} \to L(X)$ $[\mathbb{R} = (-\infty, \infty)]$ is a continuous operator function with values in the space $L(X)$ of the linear bounded operator $X \to X$, $Q_n \in L(X)$ $(n \in \mathbb{Z})$; $f: \mathbb{R} \to X$ is a locally summable function; $h = \{h_n\}_{n \in \mathbb{Z}}$ is a sequence of elements of X; $T = \{t_n\}_{n \in \mathbb{Z}}$ is a sequence of points satisfying the condition $t_n < t_{n+1}$ $(n \in \mathbb{Z})$, $\lim_{n \to \pm\infty} t_n = \pm\infty$.

Definition. The function $x(t)$: $\mathbb{R} \to X$ is said to be a solution of the impulsive equation (1), (2) if for $t \neq t_n$ it is continuously differentiable and satisfies (1) and for $t = t_n$ it satisfies the "jump" condition (2). Analogously, a solution of the nonhomogeneous equation (3), (4) is defined as well as a solution defined on an arbitrary interval.

We shall say that condition (H) is satisfied if the following condition holds:

$$\text{(H)} \quad \sup_{t \in \mathbb{R}} \lim_{\omega \to \infty} \frac{\nu(t, t+\omega)}{\omega} = q < \infty$$

where by $\nu(a, b)$ for $a \le b$ we have denoted the number of points t_n lying in the interval $(a, b]$ and for $a > b$ we set $\nu(a, b) = -\nu(b, a)$.

Note that condition (H) implies the existence of a constant l such that any interval on $\mathbb{R}$ of length l contains not more than $\lambda = l(q+1)$ points of the sequence T.

Let $1 \le p < \infty$ and let $\Omega \subset \mathbb{R}$ be a measurable set. By $L_p(\Omega, X)$ we shall denote the space of measurable functions $x: \Omega \to X$ for which $\int_\Omega \|x(t)\|^p \, dt < \infty$ with norm

$$\|x\|_{L_p(\Omega, X)} = \left(\int_\Omega \|x(t)\|^p \, dt \right)^{1/p}$$

By $l_p(X)$ we shall denote the space of sequences $h=\{h_n\}_{n\in\mathbb{Z}}$ $[h_n \in X\ (n\in\mathbb{Z})]$ for which $\sum_{n=-\infty}^{\infty} \|h_n\|^p < \infty$ with norm

$$\|h\|_{l_p(X)} = \left(\sum_{n=-\infty}^{\infty} \|h_n\|^p \right)^{1/p}$$

Analogously we define the spaces $L_\infty(\Omega, X)$ and $l_\infty(\Omega, X)$.

Let $g(t, s)$: $\mathbb{R}\times\mathbb{R} \to L(X)$ be a function satisfying the estimate

$$\|g(t, s)\| \le K e^{-\mu|t-s|} \qquad (t, s \in \mathbb{R}) \tag{5}$$

where $K, \mu > 0$.

Lemma 1. The operator G defined by the formula

$$Gf(t) = \int_{-\infty}^{\infty} g(t, s) f(s)\, ds$$

maps continuously $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, X)$ $(1 \le p \le \infty)$ and into $L_\infty(\mathbb{R}, X)$.

Proof. Set $F_s(t) = \|f(t-s)\|$. In view of $F_s \in L_p(\mathbb{R}, \mathbb{R})$, from the estimate

$$\|Gf(t)\| \le \int_{-\infty}^{\infty} K e^{-\mu|s|} \|f(t-s)\|\, ds$$

we obtain the inequalities

$$\begin{aligned} \|Gf\|_{L_p(\mathbb{R},X)} &\le \left\| \int_{-\infty}^{\infty} K e^{-\mu|s|} F_s\, ds \right\|_{L_p(\mathbb{R},\mathbb{R})} \\ &\le \int_{-\infty}^{\infty} K e^{-\mu|s|} \|F_s\|_{L_p(\mathbb{R},\mathbb{R})}\, ds \\ &= \frac{2K}{\mu} \|f\|_{L_p(\mathbb{R},X)} \end{aligned}$$

which imply that $Gf \in L_p(\mathbb{R}, X)$ and that the map G: $L_p(\mathbb{R}, X) \to L_p(\mathbb{R}, X)$ is continuous. The inclusion $Gf \in L_\infty(\mathbb{R}, X)$ and the continuity of the map G: $L_p(\mathbb{R}, X) \to L_\infty(\mathbb{R}, X)$ are easily proved by Hölder's inequality.

Lemma 1 is proved. ■

Lemma 2. Let condition (H) hold. Then the operator G defined by the formula

$$\tilde{G}h(t) = \sum_{n=-\infty}^{\infty} g(t, t_n) h_n \tag{6}$$

maps continuously $l_p(X)$ into $L_p(\mathbb{R}, X)$ $(1 \le p \le \infty)$ and into $L_\infty(\mathbb{R}, X)$.

Proof. Let $h=\{h_n\}_{n\in\mathbb{Z}}$ be an arbitrary sequence of $l_p(X)$. From (6) and Hölder's inequality there follow the inequalities

$$\begin{aligned}\|\tilde{G}h(t)\| &\leq \sum_{n=-\infty}^{\infty} \|g(t,t_n)\|\cdot\|h_n\| \\ &\leq \left(\sum_{n=-\infty}^{\infty}\|g(t,t_n)\|p'\right)^{1/p'}\left(\sum_{n=-\infty}^{\infty}\|h_n\|^p\right)^{1/p}\end{aligned}$$

where $1/p+1/p'=1$. Inequality (5) implies

$$\|\tilde{G}\|_{l_p(X)\to L_\infty(\mathbb{R},X)} \leq K\left(\sum_{n=-\infty}^{\infty} e^{-\mu p'|t-t_n|}\right)^{1/p'}$$

Condition (H) implies the estimate

$$\sum_{n=-\infty}^{\infty} e^{-\mu p'|t-t_n|} = \sum_{m=-\infty}^{\infty}\ \sum_{t+ml<t_n<t+(m+1)l} e^{-\mu p'|t-t_n|} \leq \frac{2\lambda}{1-e^{-\mu p' l}}$$

Using again Hölder's inequality, we obtain the inequalities

$$\begin{aligned}\|\tilde{G}h(t)\| &\leq \sum_{n=-\infty}^{\infty}\|g(t,t_n)\|\cdot\|h_n\| \\ &\leq K\sum_{n=-\infty}^{\infty} e^{-\mu|t-t_n|}\|h_n\| \\ &= K\sum_{n=-\infty}^{\infty} e^{-(\mu/p')|t-t_n|}\, e^{-(\mu/p)|t-t_n|}\|h_n\| \\ &\leq K\left(\sum_{n=-\infty}^{\infty} e^{-\mu|t-t_n|}\right)^{1/p'}\left(\sum_{n=-\infty}^{\infty} e^{-\mu|t-t_n|}\|h_n\|^p\right)^{1/p}\end{aligned}$$

Taking into account the estimate

$$\sum_{n=-\infty}^{\infty} e^{-\mu|t-t_n|} \leq \frac{2\lambda}{1-e^{-\mu l}}$$

we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\|\tilde{G}h(t)\|^p\,dt &\leq \left(\frac{2\lambda}{1-e^{-\mu l}}\right)^{p-1} K^p \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{-\mu|t-t_n|}\|h_n\|^p\,dt \\ &= \left(\frac{2\lambda}{1-e^{-\mu l}}\right)^{p-1} K^p \sum_{n=-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-\mu|t-t_n|}\,dt\right)\|h_n\|^p \qquad (7)\end{aligned}$$

In view of

$$\int_{-\infty}^{\infty} e^{-\mu|t-t_n|}\,dt \leq \frac{2}{\mu} \qquad (n\in\mathbb{Z})$$

(7) implies the inequality

$$\int_{-\infty}^{\infty} \|\tilde{G}h(t)\|^p \, dt \leq \frac{1}{\mu}\left(\frac{2\lambda}{1-e^{-\mu l}}\right)^{p-1} K^p \sum_{n=-\infty}^{\infty} \|h_n\|^p$$

Lemma 2 is proved. ■

3. MAIN RESULTS

3.1. Stationary Case

In this section we consider the case when

$$A(t) = A = \text{const} \cdot (t \in \mathbb{R}), \qquad Q_n = Q = \text{const} \cdot (n \in \mathbb{Z})$$

Theorem 1. Let the following conditions be fulfilled:

1. Condition (H) holds.
2. The operator Q has a logarithm $L_n Q = S \in L(X)$ and $AS = SA$.
3. The spectrum $\mathrm{Sp}(\Lambda)$ of the operator $\Lambda = A + qS$ has no points on the imaginary axis.

Then for any function $f \in L_p(\mathbb{R}, X)$ and any sequence $h \in l_p(X)$ the nonhomogeneous impulsive equation (3), (4) has in $L_p(\mathbb{R}, X)$ a unique solution and this solution is bounded.

Proof. From condition 3 of Theorem 1 it follows that $\mathrm{Sp}(\Lambda)$ is split into two spectral sets $\mathrm{Sp}_-(\Lambda)$ and $\mathrm{Sp}_+(\Lambda)$ $[\mathrm{Sp}(\Lambda) = \mathrm{Sp}_-(\Lambda) \cup \mathrm{Sp}_+(\Lambda)]$ of which $\mathrm{Sp}_-(\Lambda)$ lies in the left half-plane and $\mathrm{Sp}_+(\Lambda)$ in the right (Daleckii and Krein, 1974). The space X is split into a direct sum of two subspaces X_- and X_+, where $X_\pm = P_\pm X$ and

$$P_\pm = -\frac{1}{2\pi i} \oint_{\Gamma_\pm} R_\lambda \, d\lambda$$

where $\Gamma_\pm$ are contours surrounding $\mathrm{Sp}_\pm(\Lambda)$ in the respective half-planes and R_λ is the resolvent of the operators Λ. It is standard to verify that the operators $P_\pm$, A, S, and Λ commute two by two.

Introduce the operator-valued function G: $\mathbb{R}^2 \to \mathrm{L(X)}$ by the formula (Bainov *et al.*, 1988)

$$G(t, \tau) = \begin{cases} -e^{(t-\tau)A + \nu(t,\tau)} P_+ & -\infty < t \leq \tau < \infty \\ e^{(t-\tau)A + \nu(t,\tau)} P_-, & -\infty < \tau \leq t < \infty \end{cases}$$

The operator-valued function $G(t, \tau)$ has the following properties:

$$\begin{aligned}
&\frac{dG(t,\tau)}{dt} = AG(t,\tau) \qquad (t, \tau \in \mathbb{R}, \quad t \notin T, \quad t \neq \tau)\\
&G(t_j^+, \tau) - G(t_j^-, \tau) = QG(t_j^-, \tau) \qquad (\tau \neq t_j, j \in \mathbb{Z})\\
&G(t^+, t) - G(t^-, t) = I \qquad (t \notin T, \quad I = \mathrm{id}_{L(X)})\\
&G(t_j^+, t_j) - G(t_j^-, t_j) = QG(t_j^-, t_j) + I (j \in \mathbb{Z})\\
&\exists K, \mu > 0: \quad \|G(t,\tau)\| \leq K\, e^{-\mu|t-\tau|} \qquad (t, \tau \in \mathbb{R})
\end{aligned} \tag{8}$$

Consider the function $x(t)$ $(t \in \mathbb{R})$ defined by the formula

$$x(t) = \int_{-\infty}^{\infty} G(t,\tau) f(\tau)\, d\tau + \sum_{j=-\infty}^{\infty} G(t, t_j) h_j \qquad (t \in \mathbb{R}) \tag{9}$$

From Lemmas 1 and 2 it follows that $x(t) \in L_p(\mathbb{R}, X) \cap L_\infty(\mathbb{R}, X)$. In view of the properties of the operator-valued function $G(t, \tau)$ it is standard to show that $x(t)$ is a solution of the impulsive equation (3), (4) on $\mathbb{R}$.

We shall show that the solution $x(t)$ is unique. For this purpose we shall show that the unique solution $y(t)$ of the homogeneous equation (1), (2) which lies in $L_p(\mathbb{R}, X)$ is the solution $y(t) \equiv 0$. For the solution $y(t)$ the following representation is valid:

$$y(t) = e^{(t-t_0)A + \nu(t_0, t)S} y(t_0) \qquad (t, t_0 \in \mathbb{R})$$

For $p = \infty$ the uniqueness is proved in Bainov *et al.* (1988) (see Corollary 2 of Theorem 4). Let $1 \leq p < \infty$. Choose $t_0 = 0$ and denote $y(0) = y_0$, $B(t) = A + (1/t)\nu(0, t)S$. Then from condition (H) it follows that for sufficiently large values of $|t|$ the spectrum of the operator

$$B(t) = \Lambda + O\left(\frac{1}{t}\right) S \qquad (|t| \to \infty)$$

does not intersect the imaginary axis (Daleckii and Krein (1974)).

For $t \in \mathbb{R}$ the following representation is valid:

$$y(t) = e^{tB(t)} y_0$$

i.e.,

$$y(t) = e^{tB_+(t)} P_+(t) y_0 + e^{tB_-(t)} P_-(t) y_0$$

where $P_+(t)$ and $P_-(t)$ $(t \in \mathbb{R})$ are the respective complementary operators

$$P_\pm(t) = -\frac{1}{2\pi i} \oint_{\Gamma_\pm} R_\lambda(B(t))\, d\lambda$$

[$\Gamma_\pm$ are contours surrounding $\mathrm{Sp}_\pm(B(t))$ and $B_\pm(t) = B(t)P_\pm(t)$.] Since $B(t) \to \Lambda (t \to \infty)$, then by Theorem 2.2 of Daleckii and Krein (1974) the relation

$$P_\pm(t) \xrightarrow[|t|\to\infty]{} P_\pm \tag{10}$$

is valid, whence it follows that

$$B_\pm(t) \xrightarrow[|t|\to\infty]{} \Lambda P_\pm = \Lambda_\pm$$

Let $t \geq 0$. Then by Lemma 6.3 of Daleckii and Krein (1974) there exists constants N_0, $\nu_0 > 0$ such that the following estimate is valid:

$$\|e^{tB_-(t)}P_-(t)y_0\| \leq N_0\, e^{-\nu_0 t}\|P_-(t)y_0\| \tag{11}$$

From (10) it follows that $e^{tB_-(t)}P_-(t)y_0 \in L_p((0,\infty), X)$ and hence $e^{tB_+(t)}P_+(t)y_0 \in L_p((0,\infty), X)$. We shall show that $P_+y_0 = 0$. In fact, analogously to (11), we obtain that there exist constants N_1, $\nu_1 > 0$ such that for $t \geq 0$ the estimate

$$\begin{aligned}\|P_+(t)y_0\| &= \|e^{t(-B_+(t))}\, e^{tB_+(t)}P_+(t)y_0\| \\ &\leq N_1\, e^{-\nu_1 t}\|e^{tB_+(t)}P_+(t)y_0\|\end{aligned}$$

is valid, i.e.,

$$\|e^{tB_+(t)}P_+(t)y_0\| \geq \frac{1}{N_1}\, e^{\nu_1 t}\|P_+(t)y_0\| \tag{12}$$

Let $\varepsilon > 0$ be arbitrarily chosen. There exists a number $T(\varepsilon)$ for which for $t \geq T(\varepsilon)$ the following inequality is valid:

$$\|P_+(t)y_0\| \geq \|P_+y_0\| - \varepsilon \tag{13}$$

From (12) and (13) there follows the inequality

$$\|e^{tB_+(t)}P_+(t)y_0\| \geq \frac{1}{N_1}\, e^{\nu_1 t}(\|P_+y_0\| - \varepsilon)$$

Since $e^{tB_+(t)}P_+(t)y_0 \in L_p((0,\infty), X)$, then $\|P_+y_0\| = 0$, i.e., $y_0 \in X_-$. Analogously, if we consider the case $t \to -\infty$, we obtain that $y_0 \in X_+$, whence we conclude that $y_0 = 0$. Hence the trivial solution is the unique solution of the homogeneous equation (1), (2) which lies in $L_p(\mathbb{R}, X)$.

Theorem 1 is proved. ■

Remark 1. From formula (9) and Lemmas 1 and 2 it follows that if $f \in L_\infty(\mathbb{R}, X)$ and $h \in l_\infty(X)$, then for $x(t)$ the following estimate is valid:

$$\|x\|_{L_\infty(\mathbb{R},X)} \leq C \max\left\{ \|f\|_{L_\infty(\mathbb{R},X)}, \sup_{j\in\mathbb{Z}} \|h_j\| \right\}$$

where the constant C does not depend on f and h.

The following theorem is in a certain sense inverse to Theorem 1.

Theorem 2. Let a number p $(1 \le p \le \infty)$ exist such that for any function $f \in L_p(\mathbb{R}, X)$ the equation

$$x' = Ax + f(t) \tag{14}$$

has just one solution $x(t) \in L_p(\mathbb{R}, X)$.

Then the operator A has no eigenvalues on the imaginary axis.

Proof. Let $y \in X$ $(y \neq 0)$ be an eigenvector of the operator corresponding to the eigenvalue $i\rho$. If $p < \infty$, then set

$$f(t) = \begin{cases} e^{i\rho t} y, & |t| \le 1 \\ 0, & |t| > 1 \end{cases} \tag{15}$$

Let $x(t)$ be the solution corresponding to the function (15) which belongs to $L_p(\mathbb{R}, X)$. We apply the operator A to both sides of (14) and obtain

$$(Ax)' \quad (= A(x')) = A(Ax) + i\rho f(t)$$

Since the solution of (14) which belongs to $L_p(\mathbb{R}, X)$ is unique, then $Ax(t) = i\rho x(t)$. Then from (14) it follows that

$$(x\, e^{-i\rho t})' = e^{-i\rho t} f(t)$$

Integrate equality (16) from $-t$ to t $(t \ge 1)$ and obtain

$$x(t)\, e^{-i\rho t} - x(-t)\, e^{i\rho t} = 2y \qquad (t \ge 1)$$

which obviously contradicts the condition $x(t) \in L_p(\mathbb{R}, X)$.

For $p = \infty$ set $f(t) = e^{i\rho t} y$ $(t \in \mathbb{R})$. Then, integrating (16) from 0 to t, we obtain

$$x(t)\, e^{-i\rho t} - x(0) = ty, \qquad -\infty < t < \infty$$

from which for $|t| \to \infty$ we again obtain a contradiction.

Theorem 2 is proved. ■

3.2. Nonstationary Case

Denote by $U(t, \tau)$ $(t, \tau \ge 0)$ the evolutionary operator of the impulsive equation (1), (2) (Zabreiko *et al.*, 1988), i.e., the linear operator which associates with an arbitrary $x_0 \in X$ the solution $x = x(t, \tau; x_0)$ of the impulsive equation (1), (2) with initial condition $x(\tau^+) = x_0$.

Theorem 3. Let the following conditions be fulfilled:

1. Condition (H) holds.
2. There exist constants N, $\nu > 0$ such that

$$\| U(t, \tau) \| \le N e^{-\nu(t-\tau)} \qquad (0 \le \tau \le t < \infty)$$

Then for any function $f \in L_p(\mathbb{R}_+, X)$ $[\mathbb{R}_+ = (0, \infty)]$ and any sequence $h \in l_p(X)$ there exists a solution of the impulsive equation (3), (4) defined on the semiaxis $\mathbb{R}_+$ and lying in $L_p(\mathbb{R}_+, X) \cap L_\infty(\mathbb{R}_+, X)$.

Proof. It is standard to verify that an arbitrary solution $x(t)$ of the impulsive equation (3), (4) defined on $\mathbb{R}_+$ is represented in the form

$$x(t) = U(t, 0)x(0^+) + \int_0^t U(t, \tau)f(\tau)\, d\tau + \sum_{0<t_n<t} U(t, t_n)h_n$$

The proof of Theorem 3 follows from condition 2 and Lemmas 1 and 2.

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